Excitation of travelling multibreathers in anharmonic chains

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Abstract

We study the dynamics of the "externally" forced and damped Fermi-Pasta-Ulam (FPU) 1D lattice. The forcing has the spatial symmetry of the Fourier mode with wavenumber p and oscillates sinusoidally in time with the frequency ω. When ω is in the phonon band, the p-mode becomes modulationally unstable above a critical forcing, which we determine analytically in terms of the parameters of the system. For ω above the phonon band, the instability of the p-mode leads to the formation of a *travelling multibreather*, that, in the low-amplitude limit could be described in terms of soliton solutions of a suitable driven-damped nonlinear Schrödinger (NLS) equation. Similar mechanisms of instability could show up in easy-axis magnetic structures, that are governed by such NLS equations.

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1 Introduction

Chains of anharmonic oscillators constitute an ideal testing ground for the different theoretical approaches to the study of the dynamics of many-degrees-of-freedom systems. For instance, the first numerical study of the *approach to equilibrium* was performed by Fermi, Pasta and Ulam (FPU), initializing the chain in a long wave-length state and looking at mode energy sharing at long-times [1]. Later on, the role of certain exact nonlinear standing and travelling wave solutions of such Hamiltonian lattices was discussed in relation with

chain excitations at shorter wavelengths [2–4]. This led to the discovery of *chaotic breathers* [5], spatially highly localized and erratically moving objects, which appear in the process of thermalization at small energy densities (these were in fact previously found in Ref. [6] without emphasizing too much their role for the approach to equilibrium). This discovery motivated a series of papers [7–9] where the formation, lifetimes and destruction of these "intrinsically localized" structures were studied in full detail. Very recently, a different approach was followed in which, instead of looking at the fate of a specific initial state of such FPU lattices, e.g. the highest frequency zone-boundary Fourier π-mode, a driving term with this spatial symmetry was directly added to the Hamiltonian equations of motion [10]. Damping was included to allow the formation of stationary states and it was taken to be weak to remain close to the Hamiltonian case. As a result the zone-boundary mode was found to be stable and locked to the driving field below a certain critical forcing, which is indeed analytically calculable. Above such forcing, a *standing modulated wave* forms for driving frequencies that are below the band edge, while a *static multibreather* (a train of breathers) develops for forcing frequencies above the band edge. The patterns emerging from the instability of the zone-boundary mode are always almost static, because the group velocity of the carrier wave is zero. They can acquire a nonzero velocity due to other nonlinear mechanisms [7].

In this paper we study a damped FPU 1D lattice forced by a travelling wave field with wavenumber $p \neq \pi$. An instability mechanism similar to the one mentioned above leads to the formation of *travelling modulated waves* for in-band frequencies. When the forcing frequency is instead above the phonon band, a *moving multibreather* arises after instability of the p-mode. This state can indeed be thought as a moving envelope soliton train (the carrier wave having the symmetry of the p-mode), since the envelope of these "intrinsically localized" moving objects can be put into a close relation with the soliton solutions of a suitable externally driven and damped nonlinear Schrödinger (NLS) equation. Such type of equation arises in the small amplitude limit of the ac driven damped sine-Gordon (SG) system; it is discussed in the literature and its analytical and numerical solutions are known [11–13].

In Section 2 we introduce a Fourier mode representation of the FPU lattice and we study it in the *rotating wave* approximation and for weak damping, obtaining an explicit analytical expression for the critical amplitude f_{cr} in terms of the parameters of the system. Section 3 is devoted to the description of the travelling multibreather, both numerically and, analytically, in terms of a suitable NLS equation. In Section 4 we draw some conclusions.

2 Stability of the travelling wave

The equations of motion of the "externally driven" damped FPU chain read as follows:

$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n + (u_{n+1} - u_n)^3 + (u_{n-1} - u_n)^3 \\ -\gamma \dot{u}_n + f \cos(\omega t + pn), \qquad (1)$$

where u_n is the displacement of n-th oscillator with respect to its equilibrium position. Periodic boundary conditions, $u_{n+N} = u_n$, are assumed, with N being the number of oscillators. Dimensionless units are used such that the masses, the linear and nonlinear force constants and the lattice spacing are taken equal to unity. The forcing and damping strengths are gauged by the parameters f and γ, respectively; ω and p are the driving frequency and wavenumber. For technical reasons, to be discussed below, we restrict to $\pi/2 < |p| < \pi$. As in Ref. [10] we use the following expansion:

$$u_n = \frac{1}{2} \sum_k \left[a_k \, e^{i(\omega t + kn)} + a^+_{-k} \, e^{-i(\omega t - kn)} \right], \qquad (2)$$

where a_k are complex mode amplitudes and $-\pi < k \leq \pi$ the corresponding wavenumbers. The equations of motion for the amplitudes a_k's are obtained by substituting Eq. (2) into Eq. (1). Similarly to what is done for the undamped case [14], a considerable simplification is achieved by neglecting higher-order harmonics that are produced by the cubic force terms. This is the so-called *rotating wave approximation*, that is valid for small amplitudes, $|a_k| \ll 1$. Moreover, in the limit of weak damping $\gamma \ll \omega$ and neglecting inertial terms $\ddot{a}_k$, we find the following set of approximate equations:

$$-2i\omega \dot{a}_k - i\omega\gamma a_k = (\omega_k^2 - \omega^2) a_k + f\delta_{k,p} + 6 \sum_{q_1,q_2} G^k_{q_1,q_2} a_{q_1} a_{q_2} a^+_{q_1+q_2-k}, \qquad (3)$$

where

$$\omega_k^2 = 2(1 - \cos k), \qquad (4)$$

$\delta_{k,p}$ is equal to one for $k = p$ and zero otherwise and

$$G^k_{q_1,q_2} = \frac{1}{4}[1 + \cos(q_1 + q_2) + \cos(k - q_2) + \cos(k - q_1) \\ - \cos k - \cos q_1 - \cos q_2 - \cos(k - q_1 - q_2)]. \qquad (5)$$

Let us begin by considering solutions where only the p-mode is excited. Such solutions, corresponding to nonlinear travelling waves with velocity $v = \omega/p$, are numerically observed to exist and to be stable below a certain critical forcing f_{cr}. Indeed, when we initialize the lattice at equilibrium and we impose random velocities, all modes with $k \neq p$ damp out on a time-scale set by the value of γ^{-1} while $|a_p|$ rapidly grows and finally approaches a constant value. In this asymptotic regime the system is well described by the "internal dynamics" of the p-mode, given by the following equation:

$$-2i\omega\dot{a}_p - i\omega\gamma a_p = (\omega_p^2 - \omega^2)a_p + f + \frac{3}{4}\omega_p^4 a_p |a_p|^2 \ . \tag{6}$$

The asymptotic amplitude a_p is then obtained by solving for the stationary solution of Eq. (6), which originates the algebraic equation below:

$$a_p = \frac{f}{\omega^2 - \omega_p^2 - \frac{3}{4}\omega_p^4 |a_p|^2 - i\gamma\omega}, \tag{7}$$

which can be numerically solved for both the modulus and the phase of a_p. In particular the squared modulus of a_p, $z = |a_p|^2$, is the solution of the cubic equation:

$$\frac{9}{16}\omega_p^8 z^3 - \frac{3}{2}\omega_p^4(\omega^2 - \omega_p^2)z^2 + \left[(\omega^2 - \omega_p^2)^2 + \gamma^2\omega^2\right] z = f^2 \quad . \tag{8}$$

In reality, also the higher odd harmonics of p will be excited, but they can be neglected when $|a_p| \ll 1$. One can easily ascertain, as it also happens for the $p = \pi$ case [10], that Eq. (8) admits a single real root only for $|\omega| < \omega_*$ where

$$\omega_* \simeq \omega_p + \sqrt{3}\gamma/2, \tag{9}$$

while three distinct roots may otherwise exist. We will not analyse in this short note what happens when three solutions of the consistency equation (7) coexist for a given parameter set. It turns out that, the destabilization of the p-mode can be described by considering the lowest amplitude solution of (6).

The p-mode nonlinear solution is stable for $f < f_{cr}(\omega, \gamma, p)$. It is possible to get an analytic expression for $f_{cr}(\omega, \gamma, p)$ in the range $0 < \omega < \omega_*$. This is accomplished by solving the following set of equations, that one obtains when linearizing Eqs. (3) around the stationary p-mode solutions (i.e. neglecting all mode interaction terms which are nonlinear in the a_k's)

$$-2i\omega\dot{a}_k - i\omega\gamma a_k = (\tilde{\omega}_k^2 - \omega^2)a_k + \frac{3}{2}\omega_p^2 B_{k,p} a_p^2 a_{2p-k}^+$$

$$-2i\omega\dot{a}^+_{2p-k} - i\omega\gamma a^+_{2p-k} = -(\tilde{\omega}^2_{2p-k} - \omega^2)a^+_{2p-k} - \frac{3}{2}\omega_p^2 B_{k,p}(a_p^+)^2 a_k, \qquad (10)$$

where

$$\tilde{\omega}_k^2 = (1 + \frac{3}{2}\omega_p^2|a_p|^2)\omega_k^2, \qquad B_{k,p} = \cos(k-p) - \cos p \qquad (11)$$

are the frequency of the k-th mode shifted by the interaction with the p-mode and $B_{k,p}$ derive from $G^k_{q_1,q_2}$ in Eq. (5), when the interaction is restricted to the mode subset $k, p, 2p-k$. Since $B_{k,p} = B_{2p-k,p}$, the two Eqs. (10) can be obtained one from the other for $k \to (2p-k)$, hence they can be solved looking for symmetric solutions of the form $a_k \sim a_{2p-k} \sim \exp(\nu_k t)$. The relevant branch of the eigenvalue spectrum reads

$$\begin{aligned}\nu_k = &-\frac{\gamma}{2} + i\frac{\tilde{\omega}_k^2 - \tilde{\omega}^2_{2p-k}}{4\omega} \\ &+\frac{1}{2|\omega|}\sqrt{\frac{9}{4}\omega_p^4 B^2_{k,p}|a_p|^4 - \left(\omega^2 - \frac{1}{2}(\tilde{\omega}_k^2 + \tilde{\omega}^2_{2p-k})\right)^2}\ . \end{aligned} \qquad (12)$$

The growth rate $Re\{\nu_k\}$ is maximal when the square root in the above expression attains its maximum value, i.e. when the "resonance" condition

$$2\omega^2 = \tilde{\omega}_k^2 + \tilde{\omega}^2_{2p-k} \qquad (13)$$

holds. The latter, together with the expression for $\tilde{\omega}_k^2$ in (11), allows to get the following equation for the wavenumber k_* of the most unstable mode

$$\cos(p - k_*) = \frac{1}{\cos p}\left(1 - \frac{\omega^2}{2 + 3\omega_p^2|a_p|^2)}\right)\ , \qquad (14)$$

in terms of the amplitude of the p-mode. Moreover, if we let $Re\{\nu_k\} = 0$ in Eq. (12) and use the definition of $B_{k,p}$ in (11), we obtain the following equation for the squared amplitude of the p-mode:

$$|a_p|^2 = \frac{2\gamma\omega}{3\omega_p^2|\cos(k_* - p) - \cos p|}\ . \qquad (15)$$

Solving the set of Eqs. (14) and (15) for both k_* and $|a_p|$ one obtains the critical amplitude $|a_p|_{cr}$ of the p-mode, above which modulational instability occurs. Finally, replacing the expression for $z_{cr} = |a_p|^2_{cr}$ in Eq. (8), we obtain $f_{cr}(\gamma, \omega, p)$, as announced above. We do not display its explicit expression because of the rather bulky form. All this is of course valid when only one

solution of the cubic equation (8) is present, i.e. for $\omega < \omega_*$. Moreover, Eq. (14) has a solution in this full ω range only when $\pi/2 < |p| < \pi$, as can be easily checked. In this range of parameters we can get rather general expressions both for the stable p-mode pattern and for the expression of the critical forcing f_{cr}.

When $f > f_{cr}$ the p-mode is modulationally unstable and, presumably, as for the $p = \pi$ case [10], the system saturates asymptotically, due to nonlinear effects, into a state where the triplet of modes $k_*, p, 2p - k_*$ is excited, resulting into a more complex *travelling modulated wave*. Its expression can possibly be computed, solving for the stationary solution of the coupled equations for this triplet of modes. This calculation is out of the scope of the present short note.

3 Travelling multibreathers

As it was mentioned above, in the range $|\omega| > \omega_* = \omega_p + \sqrt{3}\gamma/2$, three roots of Eq. (8) are present and the instability process is determined by the "internal" dynamics of the p-mode, as defined by Eq. (6). After instability, several modes grow at the same time and no periodic pattern develops. However, numerical simulations show that a *travelling multibreather* can form just above threshold. An example is shown in Fig. 1, where we plot the local energy

$$h_n = \frac{1}{2}\dot{u}_n^2 + \frac{1}{2}\left[(u_{n+1} - u_n)^2 + (u_n - u_{n-1})^2\right] + \frac{1}{4}\left[(u_{n+1} - u_n)^4 + (u_n - u_{n-1})^4\right] \ , \tag{16}$$

vs. the lattice position sampled at the period of the forcing. The corresponding spatial Fourier spectrum is shown in Fig. 2. The broad band structure of the spectrum reflect the non perfect periodic arrangement of the localized peaks in Fig. 1.

Such states can be described in terms of soliton solutions of an associated suitable driven-damped nonlinear Schrödinger (NLS) equation. Let us first make the following definition:

$$u_n = \frac{1}{2}\left[a_p(n,t)e^{i(\omega t + pn)} + a_p^+(n,t)e^{-i(\omega t + pn)}\right], \tag{17}$$

where a_p and its conjugate are smooth functions of n. Such an assumption is possible if the wavepacket is concentrated around the driving mode $\Delta k \ll \pi$. Substituting (17) into the equations of motion (1) one gets:

$$
\begin{aligned}
(\omega_p^2 - \omega^2) a_p &+ 2i\omega \left(\frac{\partial a_p}{\partial t} - v \frac{\partial a_p}{\partial n} \right) \\
&+ \frac{\omega_p^2}{4} \frac{\partial^2 a_p}{\partial n^2} + \frac{3}{4} \omega_p^4 a_p |a_p|^2 = -i\omega\gamma a_p - f \ . \qquad (18)
\end{aligned}
$$

After performing the following re-scalings:

$$
t' = \frac{\omega^2 - \omega_p^2}{2\omega} t, \qquad \xi = \frac{2\sqrt{\omega^2 - \omega_p^2}}{\omega_p} (n - vt),
$$

$$
\Psi = \sqrt{\frac{3}{8}} \frac{\omega_p^2}{\sqrt{\omega^2 - \omega_p^2}} e^{it'} a_p(\xi, t'), \qquad \gamma' = \frac{\omega}{\omega^2 - \omega_p^2} \gamma,
$$

$$
h = \sqrt{\frac{3}{8}} \frac{\omega_p^2}{(\omega^2 - \omega_p^2)^{3/2}} f \ ,
$$

and choosing a reference frame moving with velocity $v = \partial \omega_p / \partial p = \sin p / \omega_p$, Eq.(18) reduces to the well studied "externally" driven (or ac driven) damped NLS equation [11,12]:

$$
i \frac{\partial \Psi}{\partial t'} + \frac{\partial^2 \Psi}{\partial \xi^2} + 2\Psi |\Psi|^2 = -i\gamma' \Psi - h e^{it'} \ . \qquad (19)
$$

Exact soliton solutions of this equation can be obtained for $\gamma' = 0$, see Eqs.(37-40) of Ref. [11]. Moreover, multisoliton solutions are also derived in Ref. [12]. What we observe in Fig. 1 might well be a superposition of such solutions to form a train of "intrinsically localized" structures. However, one should bear in mind that NLS solutions can describe only low amplitude states. Therefore, they can be only a fair approximation of the pattern displayed in Fig. 1, which shows high amplitude localized peaks.

In Fig. 3 we plot the speed of the travelling multibreather as a function of the wavenumber of the forcing p-mode, which compares well with the group velocity of the corresponding linear waves, showing that nonlinear effects are negligible in this parameter range.

4 Conclusions

We have shown how to extend the analysis of the π-mode solution in an "externally" forced and damped FPU 1D lattice performed in Ref. [10] to that of a generic p-mode solution with $\pi/2 < |p| < \pi$. Since the carrier wave has a nonzero velocity, we find, for weak forcing, solutions corresponding to *nonlinear travelling waves*. These solutions become unstable above a critical forcing

f_{cr}, of which an analytical expression can be derived in terms of the parameters of the system: the frequency of the forcing ω, the damping constant γ and the wavenumber p. Above f_{cr} the p-mode solution is modulationally unstable and the system generates complex patterns after nonlinear saturation: *travelling modulated waves* for in-band frequency forcing $\omega < \omega_*$ or *travelling multibreathers* for out-band forcing $\omega > \omega_*$. We suggest that the multibreather pattern could be described by the soliton solutions of a driven-damped nonlinear Schrödinger equation. Travelling soliton trains have been also recently observed for the weakly damped parametrically driven NLS equation [15]. Our study also hints at the existence of similar mechanisms for the generation of moving breathers in more complex physical systems, which are known to be governed by the same driven-damped NLS or Sine-Gordon equation, e.g. one dimensional easy-axis magnetic structures [16].

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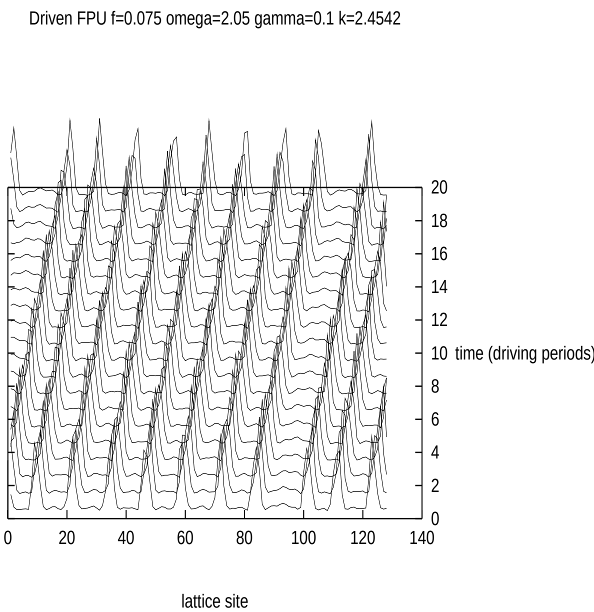

Fig. 1. Travelling multibreather pattern generated after the modulational instability of the p-mode for a lattice of $N = 512$ sites. Here, $\omega = 2.05,\ f = 0.075,\ p = 2.4542$.

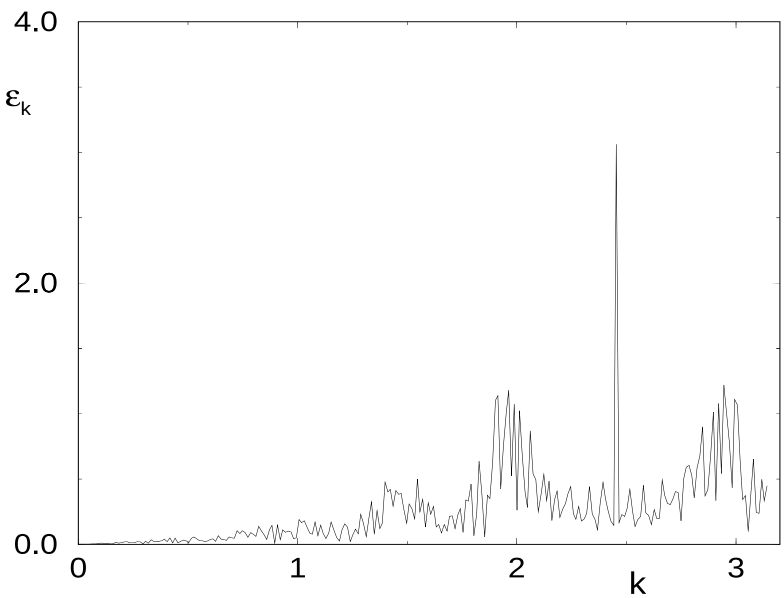

Fig. 2. Spatial spectrum of the travelling multibreather. $\epsilon_k = |\dot{U}_k| + \omega^2 |U_k|$, where U_k is the k-th component of the Fourier spectrum of the displacement field u_n. Same parameters as in Fig. 1.

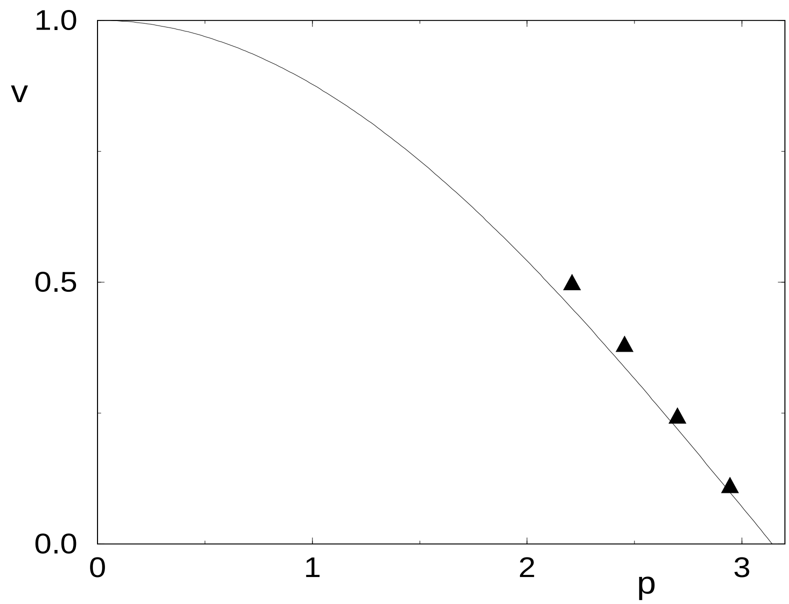

Fig. 3. Velocity of travelling multibreathers versus the wavenumber of the forcing. The solid line is the group velocity of linear waves.